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2008 J. Phys. A: Math. Theor. 41 372003

(http://iopscience.iop.org/1751-8121/41/37/372003)

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IOP Publishing — Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **41** (2008) 372003 (11pp) doi:10.1088/1751-8113/41/37/372003

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On particle creation in the flat FRW chart of de Sitter spacetime

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Received 16 June 2008, in final form 24 July 2008
Published 13 August 2008
Online at stacks.iop.org/JPhysA/41/372003

Abstract
The conditions of the Gibbons–Hawking effect, i.e., particle production in the Friedmann–Robertson–Walker chart of the de Sitter spacetime, are revisited. For a theory with a massive scalar and a fermionic field it is shown that, if one considers the Bunch–Davies vacuum state at early times, then only in the case that the condition $mc^2/\hbar H \gg 1$ is fulfilled can one assure that a thermal spectrum of radiation at temperature $T = \hbar H/2\pi k_B$, where k_B is the Boltzmann constant, will be obtained at late times. It is pointed out that this important proviso (which is nothing else than the adiabatic condition, as we shall see), is missing in several derivations of this effect in the literature, where the thermal spectrum was obtained without imposing any restriction on the relation between the mass of the field, m and the Hubble constant, H.

PACS numbers: 03.70.+k, 04.62.+v

1. Introduction

In a celebrated paper, some years ago, Gibbons and Hawking [1] showed that a particle detector following the timelike trajectory of a Killing vector in the static chart of de Sitter space detects a thermal spectrum of radiation at temperature $T = \hbar H/2\pi k_B$, H being the Hubble constant and k_B the Boltzmann constant. This effect was reproduced, for scalar particles, using the *mode mixing technique*, that is, by calculating the β-Bogoliubov coefficient that relates two different complete orthonormal sets of mode solutions. In de Sitter space these two sets are solutions of the field equations in the static and in the Friedman–Robertson–Walker (FRW) coordinates, respectively [2–5].

Several papers have since then been published where this same result is reproduced by just using the flat FRW chart of the de Sitter space [6–9]. In these, some of the quite recent calculations, their authors consider the Bunch–Davies vacuum at early times [10], and then manage to show that at late times a thermal flux of particles is created. We will here prove that, in all these cases, when a scalar and a fermionic field are invoked (as, for instance, in [6]), the result only strictly follows provided the important condition $mc^2/\hbar H \gg 1$ is satisfied, m being the mass of the field. We will also identify in the present paper specific ingredients that are missing from those re-derivations in order to arrive at the desired result.

We will also discuss the application of the WKB approximation to this problem, and show that, contrary to what is claimed in the literature [11–14], the thermal spectrum does not actually emerge in the complex WKB approach (CWKB). In section 5, taking into account specific masses for the potential particles that could be involved at cosmological scale, we will actually show that the relevant condition is not always fulfilled. Finally, in the appendix we carry out in detail the diagonalization procedure for fermionic particles that we apply in the bulk of the paper to the de Sitter spacetime, in order to calculate the fermionic particle production.

2. Massive Klein–Gordon field in (3 + 1)-dimensional de Sitter spacetime

The metric of the (3 + 1)-dimensional flat de Sitter spacetime can be written as $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right)$, where $a(t) = \mathrm{e}^{Ht}$, H being Hubble's constant. In terms of the conformal time $\eta \equiv -\frac{1}{Ha(t)}$, with $-\infty < \eta < 0$, we have $\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right)$. Next we consider a massive Klein–Gordon (KG) field ϕ, of mass m, coupled to the metric. Performing the change of function $\varphi = a\phi$ the canonical Hamiltonian using φ coincides with the Hamiltonian defined via the metrical stress–tensor for the field ϕ, and then we can easily apply the diagonalization method to the field φ [15, 16]. The KG equation becomes

$$\ddot{\varphi} - c^2\Delta\varphi + \frac{1}{\eta^2}\left[\frac{m^2c^4}{\hbar^2H^2} + 2(6\xi - 1)\right]\varphi = 0, \tag{1}$$

where the dot denotes derivative with respect to the conformal time and ξ is the coupling constant.

We now look for mode solutions of the form $\varphi(\eta, x_j) \equiv \varphi_k(\eta)\exp\left(\mathrm{i}\sum_{j=1}^{3} k_j x_j\right)$, with $k \equiv (k_1, k_2, k_3)$. We can write the KG equation as follows:

$$\ddot{\varphi}_k + \omega_k^2(\eta)\varphi_k = 0, \tag{2}$$

with $\omega_k(\eta) = \sqrt{\bar{k}^2 + \frac{\bar{\mathcal{M}}^2}{H^2\eta^2}}$, where we have introduced the notation $\bar{k} \equiv ck$ and $\frac{\bar{\mathcal{M}}^2}{H^2} \equiv \frac{\mathcal{M}^2}{H^2} + 2(6\xi - 1)$, being $\mathcal{M} \equiv mc^2/\hbar$.

The general solution of equation (2) can be written in terms of Hankel functions

$$\varphi_k(\eta) = \sqrt{\eta}\left[A_k H_\nu^{(2)}(|\bar{k}|\eta) + B_k H_\nu^{(1)}(|\bar{k}|\eta)\right], \tag{3}$$

with $\nu \equiv \sqrt{\frac{1}{4} - \frac{\bar{\mathcal{M}}^2}{H^2}}$. The asymptotic behavior of the Hankel functions, for $|z| \gg 1$, is well known [17, 18]

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\,\mathrm{e}^{\mathrm{i}(z - \frac{\pi\nu}{2} - \frac{\pi}{4})}\left[1 + \mathcal{O}\left(\frac{1}{z}\right)\right], \qquad -\pi < \arg(z) < 2\pi, \tag{4}$$

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}}\,\mathrm{e}^{-\mathrm{i}(z - \frac{\pi\nu}{2} - \frac{\pi}{4})}\left[1 + \mathcal{O}\left(\frac{1}{z}\right)\right], \qquad -2\pi < \arg(z) < \pi. \tag{5}$$

In order to define the vacuum state at early times we must choose the mode that satisfies [19, 20]

$$\varphi_k(\eta_0) = \frac{1}{\sqrt{2\omega_k(\eta_0)}}, \qquad \dot{\varphi}_k(\eta_0) = -\mathrm{i}\sqrt{\frac{\omega_k(\eta_0)}{2}}, \tag{6}$$

for $\eta_0 \to -\infty$. This definition unambiguously gives the modes in the recent past, and the frequency is constant and has the value $\bar{k}$ (specifically, this 'in' vacuum state corresponds to the Bunch–Davies vacuum). Then it is easy to verify that [19]

$$\varphi_k(\eta) = C\sqrt{\frac{\pi\eta}{4}}H_\nu^{(2)}(|\bar{k}|\eta), \qquad \text{with} \qquad C \equiv \mathrm{e}^{\mathrm{i}(|\bar{k}|\eta_0 - \frac{\pi\nu}{2} - \frac{\pi}{4})}. \tag{7}$$

Now, using the asymptotic expression for the Hankel functions at late cosmological times, $t \to \infty$ ($\eta \to 0^-$), we have, for $\eta \sim 0^-$ [17],

$$\varphi_k(\eta) \cong -C\sqrt{\frac{\eta}{4\pi}}\frac{\mathrm{i}}{\nu}\left[\mathrm{e}^{\mathrm{i}\pi\nu}\Gamma(1-\nu)\left(\frac{|\bar{k}|\eta}{2}\right)^{\nu} - \Gamma(1+\nu)\left(\frac{|\bar{k}|\eta}{2}\right)^{-\nu}\right]. \tag{8}$$

Since $\eta < 0$, if we take into account the domain of definition of the Hankel function (equation (5)), we may write $\eta = \mathrm{e}^{-\mathrm{i}\pi}|\eta|$ and then equation (8) turns into

$$\varphi_k(\eta) \cong -\mathrm{e}^{\mathrm{i}(-|\bar{k}\eta_0| - \frac{\pi}{4})}\sqrt{\frac{|\eta|}{4\pi}}\frac{1}{\nu}\left[\mathrm{e}^{-\mathrm{i}\frac{\pi\nu}{2}}\Gamma(1-\nu)\left(\frac{|\bar{k}\eta|}{2}\right)^{\nu} - \mathrm{e}^{\mathrm{i}\frac{\pi\nu}{2}}\Gamma(1+\nu)\left(\frac{|\bar{k}\eta|}{2}\right)^{-\nu}\right]. \tag{9}$$

Let us now consider the functions $e_{k,\pm}(\eta) \equiv \frac{1}{\sqrt{2\omega_k(\eta)}}\mathrm{e}^{\pm\mathrm{i}\int_{\eta_0}^{\eta}\omega_k(\tau)\,\mathrm{d}\tau}$. An easy calculation yields

$$e_{k,\pm}(\eta) = \frac{1}{\sqrt{2\omega_k(\eta)}}\mathrm{e}^{\mp\mathrm{i}|\bar{k}|\left(\sqrt{\eta^2 + \frac{\bar{\mathcal{M}}^2}{H^2|\bar{k}|^2}} - \sqrt{\eta_0^2 + \frac{\bar{\mathcal{M}}^2}{H^2|\bar{k}|^2}}\right)}\left|\frac{\eta}{\eta_0}\right|^{\mp\mathrm{i}\frac{\bar{\mathcal{M}}}{H}}\left(\frac{\sqrt{\eta_0^2 + \frac{\bar{\mathcal{M}}^2}{H^2|\bar{k}|^2}} + \frac{\bar{\mathcal{M}}}{H|\bar{k}|}}{\sqrt{\eta^2 + \frac{\bar{\mathcal{M}}^2}{H^2|\bar{k}|^2}} + \frac{\bar{\mathcal{M}}}{H|\bar{k}|}}\right)^{\mp\mathrm{i}\frac{\bar{\mathcal{M}}}{H}}. \tag{10}$$

It is a well-known difficulty that one cannot really speak about 'particle production at time t' in a dynamical (time-dependent) process. One can calculate amplitudes with respect to a given basis and speak about probabilities of finding particles in a given state, but many authors do not admit that this state be called a 'particle state' unless it becomes stationary. In fact, we are here trying to describe the change of amplitudes caused by the change of the basis between FRW and static situations (as advanced in the introduction). Now, trying to remain as close as possible to the physical (static) situation and then extrapolate it to the time-dependent FRW background case, we here make use of the instantaneous Hamiltonian diagonalization method of Zel'dovich and Starobinskii [21, 22]. That is, we choose a one parametric family of mode solutions to equation (2), namely $\{f_{k,\eta}(\tau), f_{k,\eta}^*(\tau)\}$, that satisfies the initial condition

$$f_{k,\eta}(\eta) = e_{k,-}(\eta), \qquad \dot{f}_{k,\eta}(\eta) = -\mathrm{i}\omega(\eta)e_{k,-}(\eta), \tag{11}$$

and consequently diagonalizes the Hamiltonian at time η. Since $\{f_{k,\eta}(\tau), f_{k,\eta}^*(\tau)\}$ is a basis of the space of solutions of equation (2), we can decompose the mode φ_k defined by equation (6) as follows:

$$\varphi_k(\tau) = \alpha_k(\eta)f_{k,\eta}(\tau) + \beta_k(\eta)f_{k,\eta}^*(\tau) \qquad \forall\tau \in (-\infty, 0), \tag{12}$$

where $\alpha_k(\eta)$ and $\beta_k(\eta)$ are the Bogoliubov coefficients that are now time-dependent due to the definition of the instantaneous basis $\{f_{k,\eta}, f_{k,\eta}^*\}$.

In particular, for $\tau = \eta$ one obtains the system

$$\begin{cases} \varphi_k(\eta) = \alpha_k(\eta)e_{k,-}(\eta) + \beta_k(\eta)e_{k,+}(\eta), \\ \dot{\varphi}_k(\eta) = -\mathrm{i}\omega_k(\eta)[\alpha_k(\eta)e_{k,-}(\eta) - \beta_k(\eta)e_{k,+}(\eta)]. \end{cases} \tag{13}$$

The β_k Bogoliubov coefficient is then given by

$$\beta_k(\eta) = \frac{1}{2}\left[\varphi_k(\eta) - \frac{\mathrm{i}}{\omega_k(\eta)}\dot{\varphi}_k(\eta)\right] e_{k,+}^{-1}(\eta), \tag{14}$$

and, since $|e_{k,+}(\eta)| = \frac{1}{\sqrt{2\omega_k(\eta)}}$, we obtain the following expression:

$$|\beta_k(\eta)|^2 = \frac{\omega_k(\eta)}{2}\left|\varphi_k(\eta) - \frac{\mathrm{i}}{\omega_k(\eta)}\dot{\varphi}_k(\eta)\right|^2, \tag{15}$$

or

$$|\beta_k(\eta)|^2 = \frac{1}{\hbar\omega_k(\eta)}\frac{1}{2}\left[\hbar|\dot{\varphi}_k(\eta)|^2 + \hbar\omega_k^2(\eta)|\varphi_k(\eta)|^2\right] - \frac{1}{2}. \tag{16}$$

From here we see that, working in the instantaneous diagonalization approach and using the analogy previously described, the 'number of produced particles at time η' is, within this context, actually equal to the energy of the mode $\varphi_k(\eta)$ at time η divided by the energy of the 'particle' at this time.

First, we consider the case $\frac{\tilde{\mathcal{M}}}{H} \gg 1$ (i.e. $\frac{\mathcal{M}}{H} \gg 1$), using now the fact that $\omega_k(\eta) \cong \frac{\tilde{\mathcal{M}}}{H|\eta|}$ when $|\eta| \ll \frac{\tilde{\mathcal{M}}}{H|\bar{k}|}$, a simple calculation yields (when $\eta \sim 0^-$)

$$\begin{aligned}\varphi_k(\eta) - \frac{\mathrm{i}}{\omega_k(\eta)}\dot{\varphi}_k(\eta) \cong \mathrm{i}\frac{H}{2\tilde{\mathcal{M}}}\varphi_k(\eta) + C\sqrt{\frac{\eta}{4\pi}}\Bigg[&\left(\frac{H}{\tilde{\mathcal{M}}} - \frac{\mathrm{i}}{\nu}\right)\mathrm{e}^{\mathrm{i}\pi\nu}\Gamma(1-\nu)\left(\frac{|\bar{k}|\eta}{2}\right)^{\nu}\\ &+ \left(\frac{H}{\tilde{\mathcal{M}}} + \frac{\mathrm{i}}{\nu}\right)\Gamma(1+\nu)\left(\frac{|\bar{k}|\eta}{2}\right)^{-\nu}\Bigg].\end{aligned} \tag{17}$$

In this situation $\nu \cong \mathrm{i}\frac{\mathcal{M}}{H}$ and equation (17) becomes

$$\varphi_k(\eta) - \frac{\mathrm{i}}{\omega_k(\eta)}\dot{\varphi}_k(\eta) \cong -\mathrm{i}C\sqrt{\frac{\eta}{4\pi}}\frac{H^2}{2\mathcal{M}^2}\mathrm{e}^{-\pi\frac{\mathcal{M}}{H}}\Gamma\left(1 - \mathrm{i}\frac{\mathcal{M}}{H}\right)\left(\frac{|\bar{k}|\eta}{2}\right)^{\mathrm{i}\frac{\mathcal{M}}{H}}, \tag{18}$$

thus

$$|\beta_k(\eta)|^2 \cong \frac{H^3}{32\pi\mathcal{M}^3}\left|\Gamma\left(1 - \mathrm{i}\frac{\mathcal{M}}{H}\right)\right|^2 \mathrm{e}^{\pi\frac{\mathcal{M}}{H}}. \tag{19}$$

Using at this point [17] $|\Gamma(1+z)|^2 = \pi z/\sin(\pi z)$, we conclude that, when $\eta \to 0^-$, the β-Bogoliubov coefficient is given by

$$|\beta_k(\eta)|^2 \cong \frac{H^2}{16\mathcal{M}^2}. \tag{20}$$

To obtain the thermal spectrum we need to use the so-called adiabatic interpretation of particle creation (or adiabatic vacuum prescription) in no-stationary FRW universes (see for details [23–26]). This method depends on the order of the WKB approximation used to define the mode functions. Here we use the zero order (the other orders give the same result), then the square of the β-Bogoliubov coefficient is given by [26]

$$|\beta_k(\eta)|^2 = |\varphi_k(\eta)\dot{e}_{k,+}(\eta) - \dot{\varphi}_k(\eta)e_{k,+}(\eta)|^2. \tag{21}$$

Inserting (9) and (10), in this formula, we obtain

$$|\beta_k(0^-)|^2 \cong \frac{H}{2\pi\mathcal{M}}\left|\Gamma\left(1 + \mathrm{i}\frac{\mathcal{M}}{H}\right)\right|^2 \mathrm{e}^{-\pi\frac{\mathcal{M}}{H}} = (\mathrm{e}^{2\pi\frac{\mathcal{M}}{H}} - 1)^{-1} = \left(\mathrm{e}^{2\pi\frac{mc^2}{\hbar H}} - 1\right)^{-1}, \tag{22}$$

in complete agreement with [6].

In the opposite case, $\frac{\tilde{\mathcal{M}}}{H} \ll 1$ (i.e. $\frac{\mathcal{M}}{H} \cong \sqrt{2-12\xi}$), we have $\nu \cong 1/2$ and we can make the approximation [18] $H_\nu^{(2)}(z) \cong \sqrt{\frac{2}{\pi z}}\,\mathrm{e}^{-\mathrm{i}(z-\pi/2)}$, then $\varphi(\eta) \cong C\sqrt{\frac{1}{2|\bar{k}|}}\,\mathrm{e}^{-\mathrm{i}(|\bar{k}|\eta-\pi/2)}$. Thus, inserting this expression into formula (15) with $\omega_k(\eta) \cong |\bar{k}|$ we conclude that

$$|\beta_k(\eta)|^2 \cong 0, \tag{23}$$

in agreement with the well-known fact that in the conformally coupled massless case, there is no particle production.

Note that this conclusion does not coincide with the corresponding one in [9]. In section 4 we will compare both results precisely, pointing out toward a flaw in the calculation in [9].

Some important remarks are in order. The first one is that the minimal coupling case $\xi = 0$ was previously studied in [8] in great detail. Our results here are in full agreement with those in this reference. Second, in the massless, conformally coupled case, the frequency $\omega_k(\eta) = |\bar{k}|$ does not depend on the conformal time and, consequently, in this situation there is no particle production. Moreover, note also that the energy of the produced particles at late times seems to diverge but this is actually not a problem, because at any finite time there is a transition from de Sitter space to a radiation-dominated universe where there exists a well-defined 'out' region. This physical situation has been carefully studied by Ford in [27], by making use of results previously obtained by Zel'dovich and Starobinsky [21] with the help of the diagonalization method. Essentially, equation (13) is solved there to first-order approximation and expression (15) is employed, so that the end result obtained in this way is finite and physically meaningful. Note, moreover, that the condition $mc^2/\hbar H \gg 1$ is equivalent to saying that the Hubble distance must be much larger than the Compton wavelength $(\hbar/mc)$, and this is nothing else than the adiabatic condition $|\dot{\omega}_k(\eta)| \ll \omega_k^2(\eta)$ (see section 3). Therefore, when this transition takes place, one is in fact calculating the number of 'out' particles from the 'in' vacuum in the adiabatic case, where the concept of particle is perfectly well defined and, furthermore, the results obtained are guaranteed to be independent of the procedure that is used in order to calculate them (namely, the diagonalization method or an alternative one).

3. Particle production in the WKB approximation

The zero-order WKB approximation is based on the formula

$$\varphi_k(\eta) \cong e_{k,-}(\eta). \tag{24}$$

Introducing this expression into (16), we get (using the diagonalization method)

$$|\beta_k(\eta)|^2 \cong \frac{\dot{\omega}_k^2(\eta)}{16\omega_k^4(\eta)} = \frac{H^2}{16\tilde{\mathcal{M}}^2}\left(\frac{\bar{k}^2 H^2\eta^2}{\tilde{\mathcal{M}}^2}+1\right)^{-3}. \tag{25}$$

$$\mathcal{N}(\eta) \equiv \sum_{n\in\mathbb{Z}} |\beta_k(\eta)|^2 \cong \frac{L}{16\pi}\frac{H^2}{\tilde{\mathcal{M}}^2}\int_{-\infty}^{\infty} \mathrm{d}z \left(\frac{z^2c^2H^2\eta^2}{\tilde{\mathcal{M}}^2}+1\right)^{-3} = \frac{L}{64c\eta}\frac{H}{\tilde{\mathcal{M}}}. \tag{26}$$

Moreover, the density of particles produced per unit volume is

$$\rho(\eta) \equiv \frac{\mathcal{N}(\eta)}{2L} \cong \frac{1}{128c\eta}\frac{H}{\tilde{\mathcal{M}}}. \tag{27}$$

In order to be able to rigorously prove the validity of this result, we need to impose the so-called WKB bounds (see, for instance, [29–31]). Here, for simplicity, we will just use the adiabatic condition, since the WKB *ansatz* (24) turns out to be a good approximation [28], as long as $|\dot{\omega}_k(\eta)| \ll \omega_k^2(\eta)$. It can be seen that, when $\tilde{\mathcal{M}}/H \gg 1$, the adiabatic condition is fulfilled at all time; however, in the opposite case, this condition only holds for $|\eta| \gg \left(\tilde{\mathcal{M}}/H\right)^{2/3} |\bar{k}|^{-1}$ and, consequently, in this last situation equation (25) is valid when $|\eta| \gg \left(\tilde{\mathcal{M}}/H\right)^{2/3} |\bar{k}|^{-1}$ only.

Now, we are interested in the CWKB method, that is, the WKB method in the complex plane [32, 29, 33, 30]. We consider equation (2) with $-\infty < \eta < \infty$, which is the non-physical case already studied in [11–14, 19]. It is clear that, since (7) has the asymptotic form (6), when $\eta \to -\infty$ as well as when $\eta \to \infty$, then there is *no* particle production at late times [19]. However in [11–14] the authors do claim that there *is* particle creation, at an average rate given by equation (22). It seems clear that this last result cannot hold. We will now point out to a gap in the derivations in these papers.

In all the mentioned papers, to obtain the final results the CWKB method is used. In the conformal coupling case, the frequency is given by $\omega_k(\eta) = \sqrt{\bar{k}^2 + (\mathcal{M}/H\eta)^2}$, and since this frequency does not have any real zero, one is faced up with an over-barrier reflection problem [30]. It is well known that, when $\mathcal{M}/H \gg 1$, the average number of particles produced in the k-mode is [34]

$$|\beta_k|^2 \cong \mathrm{e}^{-2Im\int_\gamma \omega_k(\tau)\,\mathrm{d}\tau}, \tag{28}$$

where the path in the complex plane is $\gamma \equiv \{R\,\mathrm{e}^{\mathrm{i}\theta} | 0 \leqslant \theta \leqslant \pi\}$, with $R \gg \frac{\mathcal{M}}{H|\bar{k}|}$. Here $\omega_k(\tau) = \frac{|\bar{k}|}{\tau}\sqrt{\tau^2 + (\mathcal{M}/Hk)^2}$ and the cut is the interval $\left[-\mathrm{i}\frac{\mathcal{M}}{H|\bar{k}|}, \mathrm{i}\frac{\mathcal{M}}{H|\bar{k}|}\right]$. Now, using the residue theorem, we can deform this path into the following one:

$$\gamma' \equiv [-R, -\epsilon] \cup \left[-\epsilon, -\epsilon + \mathrm{i}\frac{\mathcal{M}}{H|\bar{k}|}\right] \cup \left\{\mathrm{i}\frac{\mathcal{M}}{H|\bar{k}|} + \epsilon\mathrm{e}^{\mathrm{i}\theta}/\pi \leqslant \theta \leqslant 0\right\}$$
$$\cup \left[\epsilon + \mathrm{i}\frac{\mathcal{M}}{H|\bar{k}|}, \epsilon\right] \cup [\epsilon, R].$$

Along this last path, it becomes clear that $\mathrm{Im}\int_{\gamma'} \omega_k(\tau)\,\mathrm{d}\tau = 0$ and, as a consequence, that there is no particle production in this case, in full agreement with [19].

4. Comparison with other results

A different calculation was done in [9], where the minimal coupling case was studied and expression (22) was obtained, again without the need of imposing any restriction on the relation between $\mathcal{M}$ and H. This result also contradicts equation (23). We will discover a flaw in this derivation too.

Working with the coordinates (t, x_1, x_2, x_3) in the minimal coupling case, the KG equation becomes

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + 3H\frac{\mathrm{d}\phi}{\mathrm{d}t} + \bar{k}^2\,\mathrm{e}^{-2Ht}\phi + \mathcal{M}^2\phi = 0, \tag{29}$$

which is to be compared with equation (37) of [9], namely

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + \bar{k}^2\,\mathrm{e}^{-2Ht}\phi + \mathcal{M}^2\phi = 0. \tag{30}$$

Performing a convenient change of variables, for instance $z \equiv \frac{\bar{k}}{H}\,\mathrm{e}^{-Ht}$, these two equations behave, respectively, as

$$\left(z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}}{\mathrm{d}z} + z^2 + \frac{\mathcal{M}^2}{H^2}\right)\phi = 0 \tag{31}$$

and

$$\left(z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}+z\frac{\mathrm{d}}{\mathrm{d}z}+z^2+\frac{\mathcal{M}^2}{H^2}\right)\phi=0. \tag{32}$$

Of the two, it is apparent that the wrong equation is (32), e.g., the Bessel equation whose general solution is

$$\phi_k(z)=A_kH_\nu^{(2)}(z)+B_kH_\nu^{(1)}(z), \tag{33}$$

where $\nu=\mathrm{i}\frac{\mathcal{M}}{H}$. From equations (32) and (33), result (22) easily follows. However, the argument used in [9] does not stand, since for the vacuum state at early times the function $H_\nu^{(1)}(z)$ is employed, when it is quite well known that the true vacuum state there is the Bunch–Davies vacuum [10], represented by the function $H_\nu^{(2)}(z)$.

A different way to see the flaw in [9] is to consider the (1 + 1)-dimensional case. In this situation the conformal and the minimal coupling cases coincide and, thus, result (22) is incompatible with the known result that in the massless conformal coupling case there is no particle production [19]. To summarize, if we make proper use of the KG equation, equation (31), we again obtain the same result as in section 2.

For a different situation, we now consider the conformally (or minimally) coupled fermionic field in the (1 + 1)-dimensional de Sitter spacetime. This problem was previously addressed in [7], however, there are some remarks to be posed to the derivations in that paper, too. To start, a careful calculation shows that the sign of the quasi-classical solutions $e_{k,\pm}$ cannot be right (the correct solutions being those in (10)). Also, the expansion of the Hankel function at late time in [7] should be replaced by that given in equation (9) here. Finally, the average number of produced fermions at late time is given, in [7], by the expression

$$|\beta_k(0^-)|^2=(\mathrm{e}^{2\pi\frac{\mathcal{M}}{H}}+1)^{-1}, \tag{34}$$

without *any* restriction on the values of $\mathcal{M}$ and H. Again, this result cannot be thus general because, in the massless case, since the field is conformally coupled, there is no particle production, in clear contradiction with equation (34), which in the massless case yields: $|\beta_k(0^-)|^2=1/2$.

We will now derive for this case the fermionic particle production rate, by making use of the diagonalization method, which interpretation precisely coincides, in the fermionic case, with the zero-order adiabatic interpretation of particle creation. Let us start with a fermionic field ψ, and consider now the field $\Psi\equiv\eta^{-1/2}\psi$. Then, the canonical Hamiltonian obtained for the field Ψ coincides with the Hamiltonian obtained via the metric and stress–tensors for the field ψ [16] and, as a consequence, the diagonalization method can easily be applied to the field Ψ. To wit, the dynamical equations are [7]

$$\mathrm{i}\hbar\dot{\Psi}_k=\hbar\begin{pmatrix}\frac{\mathcal{M}}{H\eta} & -\mathrm{i}\bar{k}\\ \mathrm{i}\bar{k} & -\frac{\mathcal{M}}{H\eta}\end{pmatrix}\Psi_k \tag{35}$$

and the general solution of (35), when $k>0$ (the case $k<0$ can be dealt with in a similar way), can be expressed in terms of Bessel functions, as

$$\Psi_k(\eta)=A\sqrt{\eta}\begin{pmatrix}J_{-\nu}(\bar{k}\eta)\\ J_{\nu^*}(\bar{k}\eta)\end{pmatrix}+B\sqrt{\eta}\begin{pmatrix}J_{\nu}(\bar{k}\eta)\\ -J_{-\nu^*}(\bar{k}\eta)\end{pmatrix}, \tag{36}$$

where $\nu\equiv\frac{1}{2}+\mathrm{i}\frac{\mathcal{M}}{H}$. The eigenfunctions of the energy operator that appears in equation (35) are

$$v^\pm(\eta)=\frac{1}{\sqrt{2\omega_k(\eta)\left(\omega_k(\eta)\pm\frac{\mathcal{M}}{H\eta}\right)}}\begin{pmatrix}\omega_k(\eta)\pm\frac{\mathcal{M}}{H\eta}\\ \pm\mathrm{i}\bar{k}\end{pmatrix} \tag{37}$$

and, then,

$$v^{\pm}(-\infty) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm\mathrm{i}\end{pmatrix}, \qquad v^{+}(0^{-}) = \begin{pmatrix}0\\ \mathrm{i}\end{pmatrix} \qquad \text{and} \qquad v^{-}(0^{-}) = \begin{pmatrix}1\\ 0\end{pmatrix}. \tag{38}$$

When $\mathcal{M}/H \gg 1$—and only in this case—we can easily verify that the mode Ψ_k, that satisfies at early times $\Psi_k(\eta_0) = v^{+}(\eta_0)$, for $\eta_0 \to -\infty$, is

$$\Psi_k(\eta) = C\sqrt{\eta}\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\pi\nu}H_{\nu}^{(2)}(\bar{k}\eta)\\ H_{\nu^*}^{(2)}(\bar{k}\eta)\end{pmatrix}, \tag{39}$$

where $C \equiv \frac{\sqrt{\pi\bar{k}}}{2}\,\mathrm{e}^{\mathrm{i}(\bar{k}\eta_0+\frac{\pi\nu}{2}-\frac{\pi}{4})}$. Thus, in the case $\mathcal{M}/H \gg 1$, following the diagonalization method (see the appendix) when $\eta \to 0^{-}$, the square of the β-Bogoliubov coefficient turns out to be

$$|\beta_k(\eta)|^2 = \left|C\sqrt{\eta}\,\mathrm{e}^{-\mathrm{i}\pi\nu}H_{\nu}^{(2)}(\bar{k}\eta)\right|^2, \tag{40}$$

and, finally, a simple calculation yields the result

$$|\beta_k(0^{-})|^2 = (\mathrm{e}^{2\pi\frac{\mathcal{M}}{H}} + 1)^{-1}. \tag{41}$$

To finish, we prove that in the massless case there is no particle production. The field equation reads then

$$\mathrm{i}\hbar\dot{\Psi}_k = \hbar\begin{pmatrix}0 & -\mathrm{i}\bar{k}\\ \mathrm{i}\bar{k} & 0\end{pmatrix}\Psi_k \tag{42}$$

and its general solution is

$$\Psi_k(\eta) = A\begin{pmatrix}\cos(\bar{k}\eta)\\ \sin(\bar{k}\eta)\end{pmatrix} + B\begin{pmatrix}\sin(\bar{k}\eta)\\ -\cos(\bar{k}\eta)\end{pmatrix}, \tag{43}$$

with the eigenfunctions of the energy operator being

$$v^{\pm}(\eta) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm\mathrm{i}\end{pmatrix}. \tag{44}$$

By taking now

$$A \equiv \tfrac{1}{\sqrt{2}}[\cos(\bar{k}\eta_0) + \mathrm{i}\sin(\bar{k}\eta_0)], \qquad B \equiv \tfrac{1}{\sqrt{2}}[\sin(\bar{k}\eta_0) - \mathrm{i}\cos(\bar{k}\eta_0)], \tag{45}$$

we have $\Psi_k(\eta_0) = v^{+}(\eta_0)$. Finally, from the diagonalization method, we get

$$|\beta_k(\eta)|^2 = |\langle v_k^{-}(\eta), \Psi_k(\eta)\rangle|^2 = \tfrac{1}{2}|A - \mathrm{i}B|^2|\cos(\bar{k}\eta) + \mathrm{i}\sin(\bar{k}\eta)|^2 \tag{46}$$

and, as $A - \mathrm{i}B = 0$, we do obtain the announced result.

5. Conclusions

We have studied in this paper, in some detail, the issue of particle production in the flat FRW chart of the de Sitter spacetime, for a theory with a massive scalar and a fermionic field, and considering the Bunch–Davies vacuum state as the starting condition at early times.

A careful analysis has shown that the condition $mc^2/\hbar H \gg 1$ needs to be imposed if one wants to make sure that a thermal spectrum of radiation, at temperature $T = \hbar H/2\pi k_{\mathrm{B}}$—as calculated 30 years ago in a seminal work by Gibbons and Hawking [1]—will be produced. In a number of subsequent papers (some of them very recent), containing rederivations in different settings, this important condition has just been overseen, giving the impression that it was not necessary in order to obtain the Gibbons–Hawking radiation in the flat FRW chart of de Sitter spacetime, what we have shown here in detail, case by case, not to be right.

In fact, the fulfillment of this condition (whose derivation is rather non-trivial and quite elusive, as we saw in this paper) could have important consequences in applications of the Gibbons–Hawking effect at the cosmological level. In particular, it can be seen that it would really establish a clear difference, if some very light massive fields are finally proven to exist, as demanded by many theories which aim at explaining inflation and/or the current accelerated expansion of our universe. To wit, this essential condition tells us that the thermal spectrum will only be necessarily produced when the mass of the relevant field is very large as compared with the mass-equivalent of the Hubble constant. More precisely, the constraint $mc^2/\hbar H \gg 1$ is actually equivalent to say that the Hubble distance needs to be much larger than the Compton wavelength ($\hbar/mc$), and this is the adiabatic condition $|\dot{\omega}_k(\eta)| \ll \omega_k^2(\eta)$ (see section 3). Therefore, we are in fact calculating the number of particles in the adiabatic case, where the concept of particle is perfectly well defined, from the point of view of the adiabatic vacuum prescription.

A numerical calculation readily shows that the most usually predicted neutrino and axion masses, of the order of a small fraction of an eV, satisfy the above bound sufficiently well—the fraction to be compared with 1 being some 20 orders of magnitude larger. However, the bound starts to be in compromise (the fraction falls down to values of 10^8 to 10^6) for the typical masses involved in some reference CP symmetry breaking theories, as e.g. [37]. Finally, for the usual quintessence models, where in order to obtain an equation of state parameter (or barotropic index) $w \sim -1$, the scalar field mass must be extremely small—typically of the order of 10^{-33} to 10^{-27} eV—the bound is *no longer* fulfilled.

Acknowledgments

We thank the referee for interesting remarks that we took into account in this final version. The research by EE is partly based on work done while on leave at the Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, Hanover, NH 03755, USA. This investigation has been supported in part by MEC (Spain), projects MTM2005-07660-C02-01 and FIS2006-02842, and by AGAUR (Generalitat de Catalunya), contract 2005SGR-00790 and grant 2007BE-1003.

Appendix. Diagonalization method for the fermionic field

We describe here the diagonalization method for fermionic particles (the corresponding method for scalar particles has been dealt within many papers, e.g. [15, 20–22, 35, 36]). First we consider Minkowski spacetime and, to simplify, the (1+1)-dimension case. The Dirac equation is

$$\mathrm{i}\hbar\partial_t\hat{\psi} = \hat{H}\hat{\psi}, \tag{A.1}$$

and the field operator can be decomposed into Fourier modes as

$$\hat{\psi}(t,x) = \sum_{k\in\mathbb{Z}} \hat{\psi}_k(t)\frac{\mathrm{e}^{\mathrm{i}\pi kx/L}}{2L}. \tag{A.2}$$

The field operator $\hat{\psi}_k(t)$ satisfies an equation of the kind

$$\mathrm{i}\hbar\partial_t\hat{\psi}_k = \hat{H}_k(t)\hat{\psi}. \tag{A.3}$$

Let $v_k^{\pm}(t)$ be eigenvectors, with eigenvalues $\pm\hbar\omega_k(t)$, of the operator $\hat{H}_k(t)$. Then, if we decompose the field operator $\hat{\psi}_k(t)$ as

$$\hat{\psi}_k(t) = \hat{a}_k(t)\,\mathrm{e}^{-\mathrm{i}\int_{t_0}^t\omega_k}\,\mathrm{e}^{-\int_{t_0}^t\langle v_k^+,\dot{v}_k^+\rangle}v_k^+(t) + \hat{b}_k^\dagger(t)\,\mathrm{e}^{\mathrm{i}\int_{t_0}^t\omega_k}\,\mathrm{e}^{-\int_{t_0}^t\langle v_k^-,\dot{v}_k^-\rangle}v_k^-(t), \tag{A.4}$$

the Hamiltonian of the system diagonalizes

$$\mathcal{H}(t) = \sum_{k\in\mathbb{Z}} \hbar\omega_k(t)\left[\hat{a}_k^\dagger(t)\hat{a}_k(t) + \hat{b}_k^\dagger(t)\hat{b}_k(t)\right]. \tag{A.5}$$

The operators $\hat{a}_k(t)$ and $\hat{b}_k^\dagger(t)$ satisfy

$$\begin{cases} \dot{\hat{a}}_k = -\hat{b}_k^\dagger A, \\ \dot{\hat{b}}_k^\dagger = \hat{a}_k A^*, \end{cases} \tag{A.6}$$

with $A \equiv \mathrm{e}^{2\mathrm{i}\int_{t_0}^t \omega_k} \mathrm{e}^{\int_{t_0}^t \langle v_k^+, \dot{v}_k^+\rangle} \mathrm{e}^{-\int_{t_0}^t \langle v_k^-, \dot{v}_k^-\rangle} \langle v_k^+, \dot{v}_k^-\rangle$. From the canonical commutation relations and the charge conservation law it easily follows that the operators $\hat{a}_k(t)$ and $\hat{b}_k^\dagger(t)$ must be related to the Bogoliubov coefficients as

$$\begin{cases} \hat{a}_k(t) = \alpha_k(t)\hat{a}_k(t_0) + \beta_k^*(t)\hat{b}_k^\dagger(t_0), \\ \hat{b}_k^\dagger(t) = -\beta_k(t)\hat{a}_k(t_0) + \alpha_k^*(t)\hat{b}_k^\dagger(t_0). \end{cases} \tag{A.7}$$

Thus, the average number of produced particles at time t from the vacuum state at time t_0 is given by $|\beta_k(t)|^2$.

From equation (A.6), we observe that the dynamical equations for the Bogoliubov coefficients are

$$\begin{cases} \dot{\alpha}_k = \beta_k^* A, \\ \dot{\beta}_k = -\alpha_k A^*, \end{cases} \tag{A.8}$$

and, on the other hand, if we write the mode solutions as

$$\psi_k(t) = \bar{\alpha}_k(t)\,\mathrm{e}^{-\mathrm{i}\int_{t_0}^t \omega_k}\,\mathrm{e}^{-\int_{t_0}^t \langle v_k^+, \dot{v}_k^+\rangle} v_k^+(t) - \bar{\beta}_k^*(t)\,\mathrm{e}^{\mathrm{i}\int_{t_0}^t \omega_k}\,\mathrm{e}^{-\int_{t_0}^t \langle v_k^-, \dot{v}_k^-\rangle} v_k^-(t), \tag{A.9}$$

we easily see that the coefficients $\bar{\alpha}_k$ and $\bar{\beta}_k$ satisfy equations (A.8). Thus, it is clear that $\bar{\alpha}_k(t) = \alpha_k(t)$ and $\bar{\beta}_k(t) = \beta_k(t)$. Finally, we reach the conclusion that, if the mode solution satisfies $\psi_k(t_0) = v^+(t_0)$, then the average number of particles produced in the k-mode at time t is given by the expression

$$|\beta_k(t)|^2 = |\langle v_k^-(t), \psi_k(t)\rangle|^2. \tag{A.10}$$

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